

Chernoff's density is log-concave

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Abstract: We show that the density of $Z = \operatorname{argmax}\{W(t) - t^2\}$, sometimes known as Chernoff's density, is log-concave. We conjecture that Chernoff's density is strongly log-concave or "super-Gaussian", and provide evidence in support of the conjecture. We also show that the standard normal density can be written in the same structural form as Chernoff's density, make connections with L. Bondesson's class of hyperbolically completely monotone densities, and identify a large sub-class thereof having log-transforms to \mathbb{R} which are strongly log-concave.

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1. Introduction: two limit theorems

We begin by comparing two limit theorems.

First the usual central limit theorem: Suppose that X_1, \dots, X_n are i.i.d. $EX_1 = \mu$, $E(X^2) < \infty$, $\sigma^2 = \operatorname{Var}(X)$. Then, the classical Central Limit Theorem says that

$$\sqrt{n}(\bar{X}_n - \mu) \rightarrow_d N(0, \sigma^2).$$

The Gaussian limit has density

$$\phi_\sigma(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x^2}{2\sigma^2}\right) = e^{-V(x)},$$

$$V(x) = -\log \phi_\sigma(x) = \frac{x^2}{2\sigma^2} + \log(\sqrt{2\pi}\sigma)$$

$$V''(x) = (-\log \phi_\sigma)''(x) = \frac{1}{\sigma^2} > 0.$$

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Thus $\log \phi_\sigma$ is concave, and hence ϕ_σ is a *log-concave density*. As is well-known, the normal distribution arises as a natural limit in a wide range of settings connected with sums of independent and weakly dependent random variables; see e.g. [Le Cam \(1986\)](#) and [Dehling and Philipp \(2002\)](#).

Now for a much less well-known limit theorem in the setting of monotone regression. Suppose that the real-valued function $r(x)$ is monotone increasing for $x \in [0, 1]$. For $i \in \{1, \dots, n\}$, suppose that $x_i = i/(n+1)$, ϵ_i are i.i.d. with $E(\epsilon_i) = 0$, $\sigma^2 = E(\epsilon_i^2) < \infty$, and suppose that we observe (x_i, Y_i) , $i = 1, \dots, n$, where

$$Y_i = r(x_i) + \epsilon_i \equiv \mu_i + \epsilon_i, \quad i \in \{1, \dots, n\}.$$

The isotonic estimator $\hat{\underline{\mu}}$ of $\underline{\mu} = (\mu_1, \dots, \mu_n)$ is given by

$$\begin{aligned} \hat{\mu}_j &= \max_{i \leq j} \min_{k \geq j} \left\{ \frac{\sum_{l=i}^k Y_l}{k-i+1} \right\}, \\ \hat{\underline{\mu}} &= (\hat{\mu}_1, \dots, \hat{\mu}_n) \equiv T\mathbf{Y} \\ &= \text{least squares projection of } \mathbf{Y} \text{ onto } K_n, \\ K_n &= \{y \in \mathbb{R}^n : y_1 \leq \dots \leq y_n\}. \end{aligned}$$

For fixed $x_0 \in (0, 1)$ with $x_j \leq x_0 < x_{j+1}$ we set $\hat{r}_n(x_0) \equiv \hat{r}_n(x_j) = \hat{\mu}_j$.

[Brunk \(1970\)](#) showed that if $r'(x_0) > 0$ and if r' is continuous in a neighborhood of x_0 , then

$$n^{1/3}(\hat{r}_n(x_0) - r(x_0)) \rightarrow_d (\sigma^2 r'(x_0)/2)^{1/3} (2Z_1).$$

where, with $\{W(t) : t \in \mathbb{R}\}$ denoting a two-sided standard Brownian motion process started at 0,

$$\begin{aligned} 2Z_1 &= \text{slope at zero of the greatest convex minorant of } W(t) + t^2 \quad (1.1) \\ &\stackrel{d}{=} \text{slope at zero of the least concave majorant of } W(t) - t^2 \\ &\stackrel{d}{=} 2 \operatorname{argmin}\{W(t) + t^2\}. \end{aligned}$$

The density f of Z_1 is called [Chernoff's density](#). Chernoff's density appears in a number of nonparametric problems involving estimation of a monotone function:

- Estimation of a monotone regression function r : see e.g. [Ayer et al. \(1955\)](#), [van Eeden \(1957\)](#), [Brunk \(1970\)](#), and [Leurgans \(1982\)](#).
- Estimation of a monotone decreasing density: see [Grenander \(1956a\)](#), [Prakasa Rao \(1969\)](#), and [Groeneboom \(1985\)](#).
- Estimation of a monotone hazard function: [Grenander \(1956b\)](#), [Prakasa Rao \(1970\)](#), [Huang and Zhang \(1994\)](#), [Huang and Wellner \(1995\)](#).
- Estimation of a distribution function with interval censoring: [Groeneboom and Wellner \(1992\)](#), [Groeneboom \(1996\)](#).

In each case:

- There is a monotone function m to be estimated.
- There is a natural nonparametric estimator \hat{m}_n .
- If $m'(x_0) \neq 0$ and m' continuous at x_0 , then

$$n^{1/3}(\hat{m}_n(x_0) - m(x_0)) \rightarrow_d C(m, x_0)2Z_1$$

where $2Z_1$ is as in (1.1).

See [Kim and Pollard \(1990\)](#) for a unified approach to these types of problems.

The first appearance of Z_1 was in [Chernoff \(1964\)](#). Chernoff (1964) considered estimation of the mode of a (unimodal) density f via the following simple estimator: if X_1, \dots, X_n are i.i.d. with density h and distribution function H , then for each fixed $a > 0$ let

$\hat{x}_a \equiv$ center of the interval of length $2a$ containing the most observations.

Let x_a be the center of the interval of length $2a$ maximizing $H(x+a) - H(x-a) = P(X \in (x-a, x+a])$. Then Chernoff shows:

$$n^{1/3}(\hat{x}_a - x_a) \rightarrow_d \left(\frac{h(x_a + a)}{c} \right)^{1/3} 2Z_1$$

where $c \equiv h'(x_a - a) - h'(x_a + a)$. Chernoff also showed that the density $f_{Z_1} = f$ of Z_1 has the form

$$f(z) \equiv f_{Z_1}(z) = \frac{1}{2}g(z)g(-z) \quad (1.2)$$

where

$$g(t) \equiv \lim_{x \nearrow t^2} \frac{\partial}{\partial x} u(t, x),$$

where, with W standard Brownian motion,

$$u(t, x) \equiv P^{(t, x)}(W(z) > z^2, \text{ for some } z \geq t)$$

is a solution to the backward heat equation

$$\frac{\partial}{\partial t} u(t, x) = -\frac{1}{2} \frac{\partial^2}{\partial x^2} u(t, x)$$

under the boundary conditions

$$u(t, t^2) = \lim_{x \nearrow t^2} u(t, x) = 1, \quad \lim_{x \rightarrow -\infty} u(t, x) = 0.$$

Again let $W(t)$ be standard two-sided Brownian motion starting from zero, and let $c > 0$. We now define

$$Z_c \equiv \sup\{t \in \mathbb{R} : W(t) - ct^2 \text{ is maximal}\}. \quad (1.3)$$

As noted above, Z_c with $c = 1$ arises naturally in the limit theory for nonparametric estimation of monotone (decreasing) functions. [Groeneboom \(1989\)](#) (see also [Daniels and Skyrme \(1985\)](#)) showed that for all $c > 0$ the random variable Z_c has density

$$f_{Z_c}(t) = \frac{1}{2}g_c(t)g_c(-t)$$

where g_c has Fourier transform given by

$$\hat{g}_c(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda s} g_c(s) ds = \frac{2^{1/3} c^{-1/3}}{Ai(i(2c^2)^{-1/3} \lambda)}. \quad (1.4)$$

[Groeneboom and Wellner \(2001\)](#) gave numerical computations of the density f_{Z_1} , distribution function, quantiles, and moments.

Recent work on the distribution of the supremum $M_c \equiv \sup_{t \in \mathbb{R}} (W(t) - ct^2)$ is given in [Janson et al. \(2010\)](#) and [Groeneboom \(2010\)](#). [Groeneboom \(2011\)](#) studies the number of vertices of the greatest convex minorant of $W(t) + t^2$ in intervals $[a, b]$ with $b - a \rightarrow \infty$; the function g_c with $c = 1$ also plays a key role there.

Our goal in this paper is to show that the density f_{Z_c} is log-concave. We also present evidence in support of the conjecture that f_{Z_c} is strongly log-concave: i.e. $(-\log f_{Z_c})''(t) \geq \text{some } c > 0$ for all $t \in \mathbb{R}$.

The organization of the rest of the paper is as follows: log-concavity of f_{Z_c} is proved in [Section 2](#) where we also give graphical support for this property and present several corollaries and related results. In [Section 3](#) we give some partial results and further graphical evidence for strong log-concavity of $f \equiv f_{Z_1}$: that is

$$(-\log f)''(t) \geq (-\log f)''(0) = 3.4052\dots = 1/(\cdot 541912\dots)^2 \equiv 1/\sigma_0^2$$

for all $t \in \mathbb{R}$. As will be shown in [Section 3](#), this is equivalent to $f(t) = \rho(t)\phi_{\sigma_0}(t)$ with ρ log-concave. In [Section 5](#) we briefly outline some corollaries and consequences of log-concavity and strong log-concavity of f .

2. Chernoff's density is log-concave

Recall that a function h is a *Pólya frequency function of order m* (and we write $h \in PF_m$) if $K(x, y) \equiv h(x - y)$ is totally positive of order m : i.e. $\det(H_m(\underline{x}, \underline{y})) \geq 0$ for all choices of $x_1 \leq \dots \leq x_m$ and $y_1 \leq \dots \leq y_m$ where $H_m \equiv H_m(\underline{x}, \underline{y}) = (h(x_i - y_j))_{i,j=1}^m$. It is well-known and easily proved that a density f is PF_2 if and only if it is log-concave. Furthermore, h is a *Pólya frequency function* (and we write $h \in PF_\infty$) if $K(x, y) \equiv h(x - y)$ is totally positive of all orders m ; see e.g. [Schoenberg \(1951\)](#), [Karlin \(1968\)](#), and [Marshall et al. \(2011\)](#). Following [Karlin \(1968\)](#) we say that h is *strictly PF_∞* if all the determinants $\det(H_m)$ are strictly positive.

Theorem 2.1. *For each $c > 0$ the density $f_{Z_c}(x) = (1/2)g_c(x)g_c(-x)$ is PF_2 ; i.e. log-concave.*

The Fourier transform in (1.4) implies that g_c has bilateral Laplace transform (with a slight abuse of notation)

$$\hat{g}_c(z) = \int e^{zs} g_c(s) ds = \frac{2^{1/3} c^{-1/3}}{Ai((2c^2)^{-1/3} z)} \quad (2.1)$$

for all z such that $\operatorname{Re}(z) > -a_1/(2c^2)^{-1/3}$ where $-a_1$ is the largest zero of $Ai(z)$ in $(-\infty, 0)$.

To prove Theorem 2.1 we first show that g_c is PF_∞ by application of the following two results:

Theorem 2.2. (Schoenberg, 1951) *A necessary and sufficient condition for a (density) function $g(x)$, $-\infty < x < \infty$, to be a PF_∞ (density) function is that the reciprocal of its bilateral Laplace transform (i.e. Fourier) be an entire function of the form*

$$\psi(s) \equiv \frac{1}{\hat{g}(s)} = C e^{-\gamma s^2 + \delta s} s^k \prod_{j=1}^{\infty} (1 + b_j s) \exp(-b_j s) \quad (2.2)$$

where $C > 0$, $\gamma \geq 0$, $\delta \in \mathbb{R}$, $k \in \{0, 1, 2, \dots\}$, $b_j \in \mathbb{R}$, $\sum_{j=1}^{\infty} |b_j|^2 < \infty$. (For the subclass of densities, the if and only if statement holds for $1/\hat{g}$ of this form with $\psi(0) = C = 1$ and $k = 0$.)

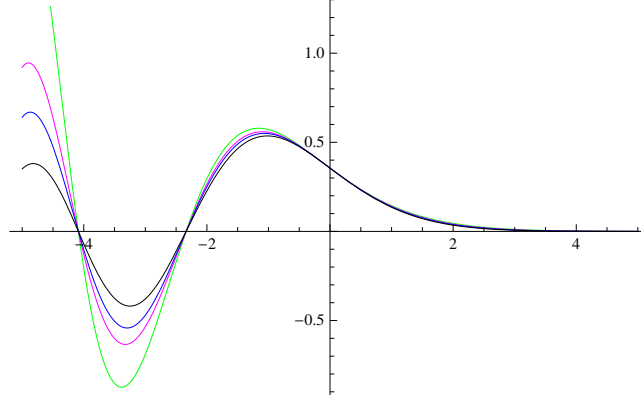
Proposition 2.1. (Merkes and Salmassi) *Let $\{-a_k\}$ be the zeros of the Airy function Ai (so that $a_k > 0$ for each k). The Hadamard representation of Ai is given by*

$$Ai(z) = Ai(0) e^{-\nu z} \prod_{k=1}^{\infty} (1 + z/a_k) \exp(-z/a_k)$$

where

$$\begin{aligned} Ai(0) &= \frac{1}{3^{2/3} \Gamma(2/3)} = \frac{\Gamma(1/3)}{3^{1/6} 2\pi} \approx 0.35503, \\ Ai'(0) &= -\frac{1}{3^{1/3} \Gamma(1/3)} = -\frac{3^{1/6} \Gamma(2/3)}{2\pi} \approx -0.25882, \text{ and} \\ \nu &= -Ai'(0)/Ai(0) = \frac{3^{1/3} \Gamma(2/3)}{\Gamma(1/3)} = \frac{2\pi}{3^{1/6} \Gamma(1/3)^2} \approx .729011 \dots \end{aligned}$$

Proposition 2.1 is given by Merkes and Salmassi (1997); see their Lemma 1, page 211. This is also Lemma 1 of Salmassi (1999). Our statement of Proposition 2.1 corrects the constants c_1 and c_2 given by Merkes and Salmassi (1997). Figure 1 shows $Ai(z)$ (black) and m term approximations to $Ai(z)$ based on Proposition 2.1 with $m = 25$ (green), 125 (magenta), and 500 (blue).

FIG 1. Product approximations of $Ai(x)$

Proposition 2.2. *The functions $t \mapsto g_c(t)$ are in $PF_\infty \subset PF_2$ for every $c > 0$. Thus they are log-concave. In fact, $t \mapsto g_c(t)$ is strictly PF_∞ for every $c > 0$.*

Proof. By Proposition 2.1,

$$\begin{aligned} Ai((2c^2)^{-1/3}z) &= Ai(0)e^{-\nu(2c^2)^{-1/3}z} \prod_{j=1}^{\infty} \left(1 + \frac{z}{(2c^2)^{1/3}a_j}\right) \exp\left(-\frac{z}{(2c^2)^{1/3}a_j}\right) \\ &= Ai(0)e^{\delta z} \prod_{j=1}^{\infty} (1 + b_j z) \exp(-b_j z) \end{aligned}$$

which is of the form (2.2) required in Schoenberg's theorem with $k = 0$,

$$\delta = -(2c^2)^{-1/3}\nu = -\frac{(3/2)^{1/3}\Gamma(2/3)}{c^{2/3}\Gamma(1/3)}, \quad (2.3)$$

$$C = Ai(0) = 1/(3^{2/3}\Gamma(2/3)), \quad \text{and} \quad (2.4)$$

$$b_j = \frac{1}{(2c^2)^{1/3}a_j}, \quad j \geq 1 \quad (2.5)$$

where $\{-a_j\}$ are the zeros of the Airy function Ai . Thus we conclude from Schoenberg's theorem that g_c is PF_∞ for each $c > 0$.

The strict PF_∞ property follows from Karlin (1968), Theorem 6.1(a), page 357: note that in the notation of Karlin (1968), $\gamma = 0$ and Karlin's a_i is our $1/a_k$ with $\sum_k (1/a_k) = \infty$ in view of the fact that $a_k \sim ((3/8)\pi(4k-1))^{2/3}$ via 9.9.6 and 9.9.18, page 18, Olver et al. (2010). \square

Now we are in position to prove Theorem 2.1:

Proof. This follows from Proposition 2.2: note that

$$-\log f_{Z_c}(x) = -\log g_c(x) - \log g_c(-x),$$

so

$$w(x) \equiv (-\log f_{Z_c})''(x) = (-\log g_c)''(x) + (-\log g_c)''(-x) \equiv v(x) + v(-x) \geq 0$$

since $g_c \in PF_\infty \subset PF_2$. \square

Some Scaling Relations: From the Fourier tranform of g_c given above, it follows that

$$\begin{aligned} g_c(x) &= \frac{(2/c)^{1/3}}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-iux}}{Ai(i(2c^2)^{-1/3}u)} du \\ &= \frac{(2/c)^{1/3}(2c^2)^{1/3}}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-iv(2c^2)^{1/3}x}}{Ai(iv)} dv \\ &\equiv 2^{1/6}c^{1/3}g_{2^{-1/2}}((2c^2)^{1/3}x). \end{aligned}$$

Thus it follows that

$$(\log g_c)''(x) = (2c^2)^{2/3} \cdot (\log g_{2^{-1/2}})''((2c^2)^{1/3}x),$$

and, in particular,

$$(\log g_c(x))'' \Big|_{x=0} = (2c^2)^{2/3} \cdot (\log g_{2^{-1/2}})''(x) \Big|_{x=0}.$$

When $c = 1$, the conversion factor is $2^{2/3}$. Furthermore we compute

$$\begin{aligned} f_{Z_c}(t) &= \frac{1}{2}g_c(t)g_c(-t) = \frac{1}{2}2^{1/3}c^{2/3}g_{2^{-1/2}}((2c^2)^{1/3}t)g_{2^{-1/2}}(-(2c^2)^{1/3}t) \\ &\equiv c^{2/3}f_1(c^{2/3}t) \end{aligned}$$

where

$$\begin{aligned} f_1(t) &\equiv f_{Z_1}(t) = \frac{1}{2}g_1(t)g_1(-t) \\ &= \frac{1}{2}2^{1/3}g_{2^{-1/2}}(2^{1/3}t)g_{2^{-1/2}}(-2^{1/3}t). \end{aligned}$$

Thus we see that

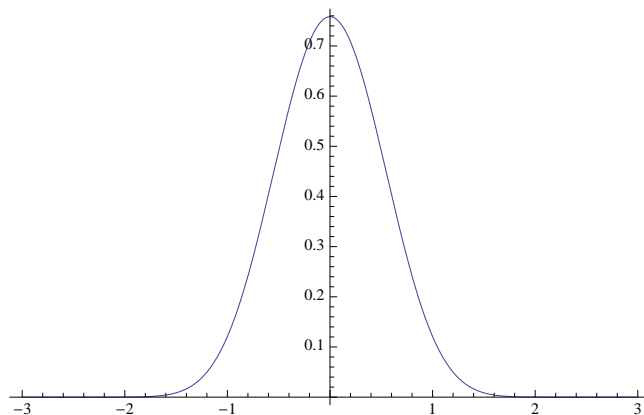
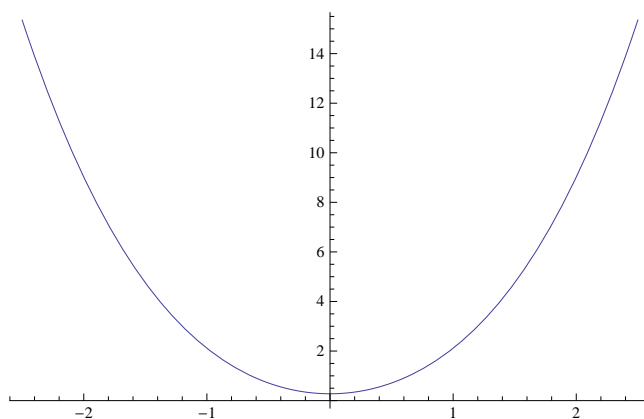
$$Z_c \stackrel{d}{=} c^{-2/3}Z_1$$

for all $c > 0$.

Figure 2 gives a plot of f_Z ; Figure 3 gives a plot of $-\log f_Z$; and Figure 4 gives a plot of $(-\log f_Z)''$.

If we use the inverse Fourier transform to represent g via (1.4), and then calculate directly, some interesting correlation type inequalities involving the Airy kernel emerge. Here is one of them.

Let $h(u) \equiv 1/|Ai(iu)| \sim 2\sqrt{\pi}u^{1/4} \exp(-(\sqrt{2}/3)u^{3/2})$ as $u \rightarrow \infty$ by Groeneboom (1989), page 95. We also define $\varphi(u, x/2) = \operatorname{Re}(e^{iux/2}Ai(iu))h(u)$ and $\psi(u, x/2) = \operatorname{Im}(e^{iux/2}Ai(iu))h(u)$.

FIG 2. The density f_Z FIG 3. $-\log f_Z$

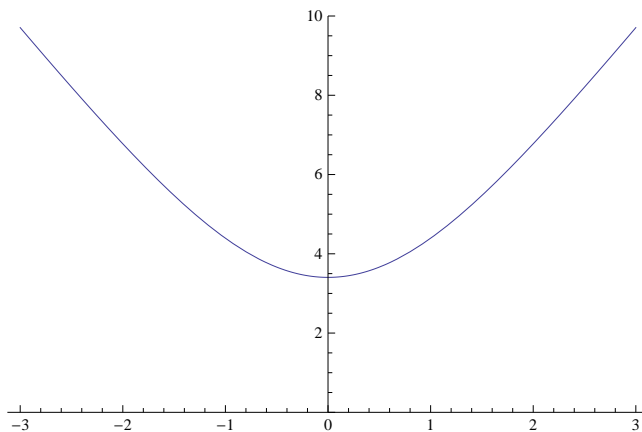
Corollary 2.1. *With the above notation,*

$$\begin{aligned} & \int_0^\infty \sin^2(uy) \varphi(u, x) h(u) du \cdot \int_0^\infty \cos^2(uy) \varphi(u, x) h(u) du \\ & + \int_0^\infty \sin(uy) \cos(uy) \psi(u, x) h(u) du \geq 0 \quad \text{for all } x, y \in \mathbb{R}. \end{aligned}$$

3. Is Chernoff's density strongly log-concave?

From [Rockafellar and Wets \(1998\)](#) page 565, $h : \mathbb{R}^d \rightarrow \overline{\mathbb{R}}$ is strongly convex if there exists a constant $c > 0$ such that

$$h(\theta x + (1 - \theta)y) \leq \theta h(x) + (1 - \theta)h(y) - \frac{1}{2}c\theta(1 - \theta)\|x - y\|^2$$

FIG 4. $(-\log f_Z)''$

for all $x, y \in \mathbb{R}^d$, $\theta \in (0, 1)$. It is not hard to show that this is equivalent to convexity of

$$h(x) - \frac{1}{2}c\|x\|^2$$

for some $c > 0$. This leads (by replacing h by $-\log f$) to the following definition of *strong log-concavity* of a (density) function: $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is strongly log-concave if and only if

$$-\log f(x) - \frac{1}{2}c\|x\|^2$$

is convex for some $c > 0$. Defining $-\log g(x) \equiv -\log f(x) - (1/2)c\|x\|^2$, it is easily seen that f is strongly log-concave if and only if

$$f(x) = g(x) \exp(-(1/2)c\|x\|^2)$$

for some $c > 0$ and log-concave function g . Thus if $f \in C^2(\mathbb{R}^d)$, a sufficient condition for strong log-concavity is: $\text{Hess}(-\log f)(x) \geq cI_d$ for all $x \in \mathbb{R}^d$ and some $c > 0$ where I_d is the $d \times d$ identity matrix.

Figure 4 provides compelling evidence for the following conjecture concerning strong log-concavity of Chernoff's density.

Theorem 3.1. (Conjectured). Let Z_1 again be a “standard” Chernoff random variable. Then for $\sigma \geq \sigma_0 \approx 0.541912\dots = (-\log f_{Z_1}(z))''|_{z=0})^{-1/2}$ the density f_{Z_1} can be written as

$$f_{Z_1}(x) = \rho(x) \frac{1}{\sigma} \varphi(x/\sigma)$$

where $\varphi(x) = (2\pi)^{-1/2} \exp(-x^2/2)$ is the standard normal density and ρ is log-concave. Equivalently, if $c \geq \sigma_0^{3/2} \approx 0.398927\dots$, then

$$f_{Z_c}(x) = \tilde{\rho}(x) \varphi(x)$$

where $\tilde{\rho}$ is log-concave.

Proof. (Partial) Let $w \equiv (-\log f_{Z_c})''$ and $v \equiv (-\log g_c)''$. Then

$$w(t) = v(t) + v(-t) \geq 2v(0) = w(0) > 0$$

is implied by convexity of v and strict positivity of $w(0)$. Thus we want to show that $v^{(2)} = (-\log g_c)^{(4)} \geq 0$.

To prove this we investigate the normalized version of g_c given by $\tilde{g}_c(x) = g_c(x)Ai(0)/(2/c)^{1/3} = g_c(x)/\int g_c(y)dy$ so that $\int \tilde{g}_c(x)dx = 1$. Suppose that b_i is given in (2.5), and let $X_i \sim \text{Exp}(1/b_i)$ be independent exponential random variables for $i = 1, 2, \dots$. Since $\sum_{i=1}^{\infty} b_i^2 < \infty$, the random variable $Y_0 = \sum_{i=1}^{\infty} (X_i - b_i)$ is finite almost surely (see e.g. Shorack (2000), Theorem 9.2, page 241) and the Laplace transform of $-(\delta + Y_0)$ is given by

$$\begin{aligned} \varphi(s) &\equiv e^{-\delta s} E e^{-s Y_0} = \exp(-\delta s) \cdot \frac{1}{\prod_{i=1}^{\infty} (1 + b_i s) e^{-b_i s}} \\ &= \frac{1}{e^{\delta s} \cdot \prod_{i=1}^{\infty} (1 + b_i s) e^{-b_i s}}, \end{aligned}$$

exactly the form of the Laplace transform in Schoenberg's theorem, but without the Gaussian term. Thus we conclude that \tilde{g}_c is the density of $Y \equiv -\delta - Y_0 = -\delta - \sum_{j=1}^{\infty} (X_j - b_j)$.

Now let $\lambda_i = 1/b_i$ for $i \geq 1$. Thus $X_i \sim \text{Exp}(\lambda_i)$. A closed form expression for the density of $Y_m \equiv \sum_{i=1}^m X_i$ has been given by Harrison (1990). From Harrison's Theorem 1, Y_m has density

$$f_m(t) = \sum_{j=1}^m \lambda_j \exp(-\lambda_j t) \prod_{i \neq j} \frac{\lambda_i}{\lambda_i - \lambda_j}. \quad (3.1)$$

If we could show that $v_m(t) \equiv (-\log f_m)''(t)$ is convex, then we would be done! Direct calculation shows that this holds for $m = 2$, but our attempts at a proof for general m have not (yet) been successful. On the other hand, we know that for $t \geq 0$,

$$w(t) = v(t) + v(-t) \geq v(t) \geq v(0) > 0$$

if v satisfies $v(t) \geq v(0)$ for all $t \geq 0$, so we would have strong log-concavity with the constant $v(0)$. \square

4. A representation of the Gaussian density as a (symmetric) product of log-concave densities

It is easily seen that the standard Gaussian density ϕ can be represented by a product of the same form as Chernoff's density (1.2) where the function g is Gaussian (and hence PF_{∞}):

$$\phi(x) = \frac{1}{2} g(x) g(-x) \quad \text{where} \quad g(x) = (8\pi)^{1/4} \phi(x/\sqrt{2}) \phi(-x/\sqrt{2}).$$

But can g be taken to be asymmetric, log-concave, and in some other interesting class?

Our goals in this section are: (a) to show that the standard Gaussian density can be written in the same product form (1.2) as Chernoff's density, but in terms of a function g that is log-concave and, in fact, is in the class of densities on \mathbb{R} given by the “log-transform” of (random variables) with densities in the class of hyperbolically completely monotone (HM_∞) densities described by Bondesson (1992, 1997); and (b) to prove that certain symmetric sub-classes of the “log-transforms” of Bondesson's hyperbolically completely monotone classes are, in fact, strongly log-concave.

The following proposition is a consequence of Bondesson (1992), Example 5.2.1.

Proposition 4.1. *The standard normal density $\phi(z) = (2\pi)^{-1/2} \exp(-z^2/2)$ can be written as*

$$\begin{aligned}\phi(z) &= \frac{1}{\sqrt{2\pi}} \exp\left(\int_0^\infty \left\{ \log\left(\frac{e^s + 1}{e^s + e^z}\right) + \log\left(\frac{e^s + 1}{e^s + e^{-z}}\right) \right\} ds\right) \quad (4.1) \\ &= \frac{1}{2} g(z) g(-z) \quad (4.2)\end{aligned}$$

where

$$\begin{aligned}g(z) &\equiv (2/\pi)^{1/4} \exp(z) \exp\left(\int_0^\infty \log\left(\frac{e^s + 1}{e^s + e^z}\right) ds\right) \\ &= (2/\pi)^{1/4} \exp\left(\frac{\pi^2}{12} + z + \int_{-e^z}^0 \frac{\log(1-t)}{t} dt\right) \quad (4.3)\end{aligned}$$

is log-concave, integrable, and $g \in \log(HM_\infty)$.

Proof. Note that

$$\begin{aligned}& \frac{1}{y} \left\{ \frac{1}{x+y} - \frac{1}{x^2} \frac{1}{y+x^{-1}} \right\} \\ &= \frac{1}{y} \left\{ \frac{x^2(y+x^{-1}) - (y+x)}{x^2(y+x)(y+x^{-1})} \right\} = \frac{x^2 - 1}{x^2(y+x)(y+x^{-1})} \\ &= \frac{x^2 - 1}{x^2(1+xy)(1+y/x)} = \frac{1}{(1+xy)(1+y/x)} - \frac{1/x^2}{(1+xy)(1+y/x)} \\ &= \frac{1}{x} \left\{ \frac{1}{y+x^{-1}} - \frac{1}{y+x} \right\}.\end{aligned}$$

Thus it follows that

$$\begin{aligned} & \int_1^\infty \left\{ \frac{1}{x+y} - \frac{1}{x^2} \frac{1}{y+x^{-1}} \right\} \frac{1}{y} dy \\ &= \frac{1}{x} \int_1^\infty \left\{ \frac{1}{y+x^{-1}} - \frac{1}{y+x} \right\} dy \end{aligned} \quad (4.4)$$

$$\begin{aligned} &= \frac{1}{x} \left\{ \log(y+1/x) - \log(y+x) \right\} \Big|_1^\infty \\ &= \frac{1}{x} \log \left(\frac{y+1/x}{y+x} \right) \Big|_1^\infty = \frac{1}{x} \left\{ 0 - \log \left(\frac{1+x^{-1}}{1+x} \right) \right\} \\ &= \frac{1}{x} \log x \end{aligned} \quad (4.5)$$

Also note that the integral on the right side in (4.4) can be rewritten as

$$\begin{aligned} & \int_1^\infty \left\{ \frac{1}{y+x^{-1}} - \frac{1}{y+x} \right\} dy = \int_1^\infty \frac{y+x-(y+x^{-1})}{(y+x^{-1})(y+x)} dy \\ &= (x-x^{-1}) \int_1^\infty \frac{1}{y^2+(x+x^{-1})y+1} dy = (x-x^{-1}) \frac{x \log x}{x^2-1} \\ &= \log x; \end{aligned}$$

this rewrite makes the integrability completely clear. Integrating the resulting identity

$$\frac{\log x}{x} = \int_1^\infty \left\{ \frac{1}{x+y} - \frac{1}{x^2} \frac{1}{y+x^{-1}} \right\} \frac{1}{y} dy$$

with respect to x on both sides over $[1, z]$ and using Fubini's theorem (permitted because the integrand is non-negative) yields

$$\begin{aligned} \frac{1}{2}(\log z)^2 &= \int_1^z \left\{ \int_1^\infty \left\{ \frac{1}{x+y} - \frac{1}{x^2} \frac{1}{y+x^{-1}} \right\} \frac{1}{y} dy \right\} dx \\ &= \int_1^\infty \left\{ \int_1^z \left\{ \frac{1}{x+y} - \frac{1}{x^2} \frac{1}{y+x^{-1}} \right\} dx \right\} \frac{1}{y} dy \\ &= \int_1^\infty \left\{ \log(y+x) + \log(y+x^{-1}) \right\} \Big|_1^z \frac{1}{y} dy \\ &= - \int_1^\infty \left\{ \log \left(\frac{y+1}{y+z} \right) + \log \left(\frac{y+1}{y+z^{-1}} \right) \right\} \frac{1}{y} dy. \end{aligned}$$

Making the change of variable of integration $y = e^s$ gives

$$\frac{1}{2}(\log z)^2 = - \int_0^\infty \left\{ \log \left(\frac{e^s+1}{e^s+z} \right) + \log \left(\frac{e^s+1}{e^s+z^{-1}} \right) \right\} ds,$$

Changing the variable z to e^x then yields the claim:

$$-\frac{1}{2}x^2 = \int_0^\infty \left\{ \log \left(\frac{e^s+1}{e^s+e^x} \right) + \log \left(\frac{e^s+1}{e^s+e^{-x}} \right) \right\} ds.$$

The claimed identity involving g follows by direct substitution. To see the second form of g , note that

$$\begin{aligned}
 \int_1^\infty \log\left(\frac{y+1}{y+z}\right) \frac{1}{y} dy &= \int_1^\infty \log\left(\frac{1+1/y}{1+z/y}\right) \frac{1}{y} dy \\
 &= \int_0^1 \log\left(\frac{1+t}{1+tz}\right) \frac{1}{t} dt \\
 &\quad \text{by the change of variables } t = 1/y \\
 &= \int_0^1 \log(1+t) \frac{1}{t} dt - \int_0^1 \log(1+tz) \frac{1}{t} dt \\
 &= \frac{\pi^2}{12} + \int_{-z}^0 \log(1-s) \frac{1}{s} ds \\
 &\quad \text{by the change of variable } -s = tz.
 \end{aligned}$$

To see that g is log-concave, note that

$$-\log g(z) = -\frac{\pi^2}{12} - z - \int_{-e^z}^0 \frac{\log(1-t)}{t} dt,$$

and hence

$$\begin{aligned}
 (-\log g)'(z) &= -1 + \frac{\log(1+e^z)}{-e^z} \cdot (-e^z) = -1 + \log(1+e^z), \\
 (-\log g)''(z) &= \frac{e^z}{1+e^z} \geq 0.
 \end{aligned}$$

Integrability of g follows from

$$-\log g(z) \sim \begin{cases} -z & \text{as } z \rightarrow -\infty, \\ -z + z^2/2 & \text{as } z \rightarrow \infty, \end{cases}$$

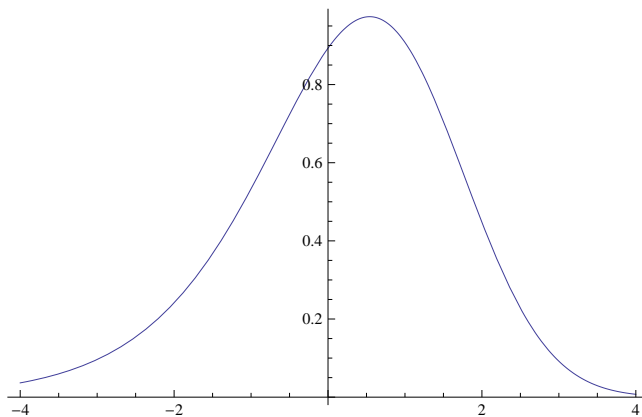
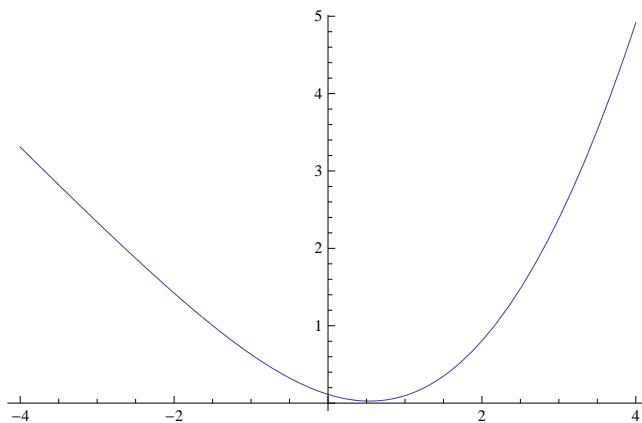
where we used

$$\int_{-y}^0 \frac{\log(1-t)}{t} dt \sim \begin{cases} 0 & \text{as } y \searrow 0, \\ -(1/2)(\log y)^2 & \text{as } y \rightarrow \infty. \end{cases}$$

Alternatively, note that $g \in \log(HM_\infty) \subset \log(HM_1) = PF_2 = \log\text{-concave}$ via [Bondesson \(1992\)](#), (5.2.3) page 73 and page 102. \square

Figures 5-8 give plots of g , $h \equiv -\log g$, and the second and third derivatives of h , namely $h^{(2)}$ and $h^{(3)}$. In fact, easy computation shows that

$$\begin{aligned}
 h^{(2)}(x) &= (-\log g)''(x) = \frac{e^x}{1+e^x}, \\
 h^{(3)}(x) &= (-\log g)^{(3)}(x) = \frac{e^x}{(1+e^x)^2},
 \end{aligned}$$

FIG 5. The function g FIG 6. The function $-\log g$

the logistic distribution function and density respectively.

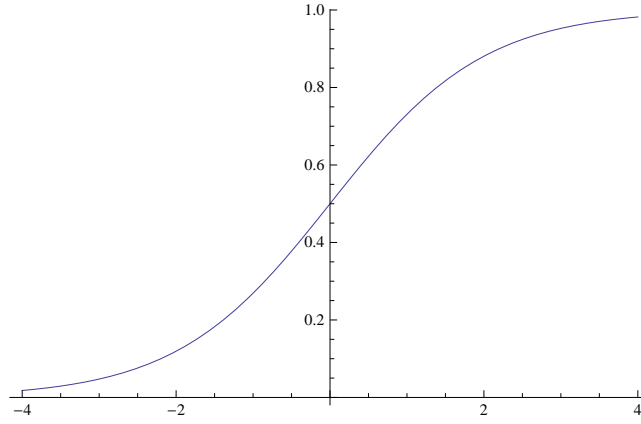
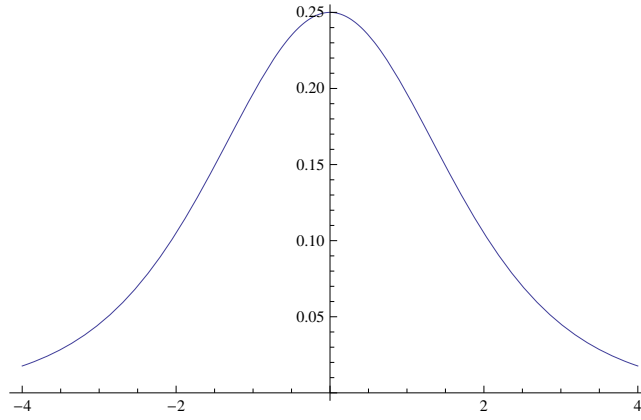
Now we examine an interesting sub-class of Bondesson's class HM_∞ which provides a rich class of strongly log-concave densities when log-transformed to \mathbb{R} . From [Bondesson \(1992\)](#), page 73, HM_∞ contains all (density) functions of the form

$$f_Y(y) = Cy^{\beta-1}h_1(y)h_2(1/y), \quad y > 0$$

where

$$h_j(y) = \exp \left(-b_j y + \int_1^\infty \log \left(\frac{v+1}{v+y} \right) d\Gamma_j(v) \right)$$

for some $b_j \geq 0$ and non-negative measures Γ_j , $j = 1, 2$, on $[1, \infty)$. Thus $X =$

FIG 7. The function $(-\log g)''$ FIG 8. The function $(-\log g)^{(3)}$

$\log Y$ has density f_X given by

$$\begin{aligned} f_X(x) &= f_Y(e^x)e^x \\ &= Ce^{(\beta-1)x}h_1(e^x)h_2(e^{-x})e^x \\ &= Ce^{\beta x}h_1(e^x)h_2(e^{-x}) = Ce^{\beta x}g_1(x)g_2(-x) \end{aligned} \quad (4.6)$$

$$= C\tilde{g}_1(x)\tilde{g}_2(-x) \quad (4.7)$$

where

$$g_j(x) \equiv h_j(e^x) = \exp\left(-b_j e^x + \int_1^\infty \log\left(\frac{v+1}{v+e^x}\right) d\Gamma_j(v)\right), \quad (4.8)$$

$$\tilde{g}_j(x) \equiv e^{\beta_j x} g_j(x), \quad (4.9)$$

$\beta_1 - \beta_2 = \beta$, and Γ_j satisfies $\int_1^\infty (1+v)^{-1} d\Gamma_j(v) < \infty$. Note that

$$\begin{aligned} -\log g_j(x) &= b_j e^x - \int_1^\infty \log\left(\frac{v+1}{v+e^x}\right) d\Gamma_j(v), \\ (-\log g_j)'(x) &= b_j e^x + \int_1^\infty \frac{e^x}{v+e^x} d\Gamma_j(v) = b_j e^x + \int_1^\infty \left(1 - \frac{v}{v+e^x}\right) d\Gamma_j(v), \\ (-\log g_j)''(x) &= b_j e^x + \int_1^\infty \frac{ve^x}{(v+e^x)^2} d\Gamma_j(v) \equiv v_j(x), \end{aligned}$$

and from this last expression we see that $v_j(x) \geq 0$; i.e. $g_1, g_2 \in PF_2$. (This is also easy and known since $\log(HM_\infty) \subset \log(HM_1) = PF_2$; see e.g. [Bondesson \(1992\)](#), p. 102.)

To show that f_X in (4.7) is strongly log-concave, we want to show that for some c

$$v_1(x) + v_2(-x) \geq c > 0 \quad \text{for all } x \geq 0$$

under some conditions on b_1, b_2 and Γ_1, Γ_2 . If $v_1 = v_2 \equiv v$, this is clearly implied by convexity of v with $c = 2v(0)$. On the other hand, since $v_j(x) \geq 0$ from log-concavity of g_j , to prove that strong log-concavity holds (perhaps with a sub-optimal constant) it suffices to show that for

$$v_j(x) \geq v_j(0) > 0 \quad \text{for all } x \geq 0 \quad \text{for either } j = 1 \text{ or } j = 2.$$

The following proposition isolates simple sufficient conditions under which strong log-concavity of f_X given by (4.7) - (4.9) holds.

Proposition 4.2. *Suppose that f_X is given by (4.7) - (4.9).*

A. *If $b_1 > 0$ and $b_2 > 0$, then f_X is strongly log-concave for any (all) measures Γ_1 and Γ_2 .*

B. *Suppose $b_1 = 0$ or $b_2 = 0$ and $d\Gamma_j(v) = v^{-1}r_j(v)dv$ for $j = 1, 2$ where (at least one of) r_1 and r_2 satisfy: (i) $r_j(y) \geq 0$ all $y \in [1, \infty)$ with strict inequality for some $y > 0$; (ii) r_j is non-decreasing. Then $v_j(x) \equiv (-\log g_j)''(x) \geq v_j(0) > 0$ for all $x \geq 0$ and hence f_X is strongly log-concave with $(-\log f_X)''(x) \geq \max\{v_1(0), v_2(0)\} > 0$.*

Proof. For part A, note that

$$\begin{aligned} (-\log f_X)''(x) &= b_1 e^x + b_2 e^{-x} + \int_1^\infty \frac{ve^x}{(v+e^x)^2} d\Gamma_1(v) + \int_1^\infty \frac{ve^{-x}}{(v+e^{-x})^2} d\Gamma_2(v) \\ &\geq b_1 e^x + b_2 e^{-x} \geq 2\sqrt{b_1 b_2} > 0 \quad \text{for all } x, \end{aligned}$$

For part B, since (at least one) Γ_j has density $\gamma_j(v) = v^{-1}r_j(v)$,

$$\begin{aligned}
(-\log g_j)''(x) &= \int_0^\infty \frac{e^{w-x}}{(1+e^{w-x})^2} e^w \gamma_j(e^w) dw \\
&= \int_0^\infty \frac{e^{w-x}}{(1+e^{w-x})^2} r_j(e^w) dw \quad \text{since } \gamma_j(y) = y^{-1}r_j(y) \\
&= \int_{-x}^\infty \frac{e^z}{(1+e^z)^2} r_j(e^{z+x}) dz \\
&= Er_j(e^{Z+x})1\{Z \geq -x\} \equiv v_j(x)
\end{aligned}$$

where $Z \sim$ standard logistic with density $e^z/(1+e^z)^2$. Then it follows that

$$\begin{aligned}
v_j(x) &= Er_j(e^{Z+x})1\{Z+x \geq 0\} = Er_j(e^{Z+x})1_{[-x,0]}(Z) + Er_j(e^{Z+x})1_{(0,\infty)}(Z) \\
&\geq Er_j(e^{Z+x})1_{(0,\infty)}(Z) \geq Er_j(e^Z)1_{(0,\infty)}(Z) = v_j(0) > 0.
\end{aligned}$$

Thus $(-\log f_X)''(x) = v_1(x) + v_2(-x) \geq v_1(0) + 0 = v_1(0)$ if the hypothesis holds for $j = 1$, while $(-\log f_X)''(x) = v_1(x) + v_2(-x) \geq 0 + v_2(0) = v_2(0)$ if the hypothesis holds for $j = 2$. Together these imply the claimed inequality. \square

Example. Note that $r(y) = (\log y)^c$ with $c > 0$ satisfies the hypotheses of the proposition. Furthermore, this choice with $c = 0$ yields the standard normal density. Note that in this case we have

$$v(x) = (-\log g)''(x) = E(Z+x)^c 1\{Z \geq -x\} \sim x^c \quad \text{as } x \rightarrow \infty.$$

5. Consequences of log-concavity and strong log-concavity of f

Log-concavity of Chernoff's density implies that the peakedness results of [Proschan \(1965\)](#) and [Olkin and Tong \(1988\)](#) apply. See also [Marshall and Olkin \(1979\)](#), page 373, and [Marshall et al. \(2011\)](#).

Note that the conclusion of the conjectured Theorem 3.1 is exactly the form of the hypothesis of the inequality of [Hargé \(2004\)](#) and of Theorem 11, page 559, of [Caffarelli \(2000\)](#); see also [Barthe \(2006\)](#), Theorem 2.4, page 1532. Another implication is that a theorem of [Caffarelli \(2000\)](#) applies: the transportation map $T = \nabla\varphi$ is a contraction. In our particular one-dimensional special case the transportation map T satisfying $T(X) \stackrel{d}{=} Z$ for $X \sim N(0,1)$ is just the solution of $\Phi(z) = F_Z(T(z))$, or equivalently $T(z) = F_Z^{-1}(\Phi(z))$. This function is apparently connected to another question concerning convex ordering of F_Z and $\Phi(\cdot)$ in the sense of [van Zwet \(1964b\)](#); see also [van Zwet \(1964a\)](#): is $T^{-1}(w) = \Phi^{-1}(F_Z(w))$ convex for $w > 0$?

6. Problems remaining

The structure of the standard normal density ϕ given in (4.2) and (4.3) is exactly the same as that of Chernoff's density (1.2) where g has Fourier transform given

in (1.4). In this case we know from Section 2 that $g \in PF_\infty$. Two natural questions are: (a) Does the function g in (1.2) satisfy $g \in \log(HM_\infty)$? (b) Does the function g in (4.3) satisfy $g \in PF_\infty$?

A further question remaining from Section 4.2: Is Chernoff's density strongly log-concave? A whole class of further problems involves replacing the (ordered) convex cone K_n in Section 1 by the convex cone \tilde{K}_n corresponding to a convexity restriction as in section 2 of Groeneboom et al. (2001b). In this latter case the limiting distribution has only been described in terms of two-sided Brownian motion; see Groeneboom et al. (2001a,b).

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References

- AYER, M., BRUNK, H. D., EWING, G. M., REID, W. T. and SILVERMAN, E. (1955). An empirical distribution function for sampling with incomplete information. *Ann. Math. Statist.* **26** 641–647.
- BARTHE, F. (2006). The Brunn-Minkowski theorem and related geometric and functional inequalities. In *International Congress of Mathematicians. Vol. II*. Eur. Math. Soc., Zürich, 1529–1546.
- BONDESSON, L. (1992). *Generalized gamma convolutions and related classes of distributions and densities*, vol. 76 of *Lecture Notes in Statistics*. Springer-Verlag, New York.
- BONDESSON, L. (1997). On hyperbolically monotone densities. In *Advances in the theory and practice of statistics*. Wiley Ser. Probab. Statist. Appl. Probab. Statist., Wiley, New York, 299–313.
- BRUNK, H. D. (1970). Estimation of isotonic regression. In *Nonparametric Techniques in Statistical Inference (Proc. Sympos., Indiana Univ., Bloomington, Ind., 1969)*. Cambridge Univ. Press, London, 177–197.
- CAFFARELLI, L. A. (2000). Monotonicity properties of optimal transportation and the FKG and related inequalities. *Comm. Math. Phys.* **214** 547–563.
- CHERNOFF, H. (1964). Estimation of the mode. *Ann. Inst. Statist. Math.* **16** 31–41.
- DANIELS, H. E. and SKYRME, T. H. R. (1985). The maximum of a random walk whose mean path has a maximum. *Adv. in Appl. Probab.* **17** 85–99.
- DEHLING, H. and PHILIPP, W. (2002). Empirical process techniques for dependent data. In *Empirical process techniques for dependent data*. Birkhäuser Boston, Boston, MA, 3–113.
- GRENANDER, U. (1956a). On the theory of mortality measurement. I. *Skand. Aktuarietidskr.* **39** 70–96.
- GRENANDER, U. (1956b). On the theory of mortality measurement. II. *Skand. Aktuarietidskr.* **39** 125–153 (1957).
- GROENEBOOM, P. (1985). Estimating a monotone density. In *Proceedings of the Berkeley conference in honor of Jerzy Neyman and Jack Kiefer, Vol. II*

- (Berkeley, Calif., 1983). Wadsworth Statist./Probab. Ser., Wadsworth, Belmont, CA.
- GROENEBOOM, P. (1989). Brownian motion with a parabolic drift and Airy functions. *Probab. Theory Related Fields* **81** 79–109.
- GROENEBOOM, P. (1996). Lectures on inverse problems. In *Lectures on probability theory and statistics (Saint-Flour, 1994)*, vol. 1648 of *Lecture Notes in Math.* Springer, Berlin, 67–164.
- GROENEBOOM, P. (2010). The maximum of Brownian motion minus a parabola. *Electronic Journal of Probability* **15** 1930–1937.
- GROENEBOOM, P. (2011). Vertices of the least concave majorant of Brownian motion with parabolic drift. *Electronic Journal of Probability* **16** 2234–2258.
- GROENEBOOM, P., JONGBLOED, G. and WELLNER, J. A. (2001a). A canonical process for estimation of convex functions: the “invelope” of integrated Brownian motion $+t^4$. *Ann. Statist.* **29** 1620–1652.
- GROENEBOOM, P., JONGBLOED, G. and WELLNER, J. A. (2001b). Estimation of a convex function: characterizations and asymptotic theory. *Ann. Statist.* **29** 1653–1698.
- GROENEBOOM, P. and WELLNER, J. A. (1992). *Information bounds and nonparametric maximum likelihood estimation*, vol. 19 of *DMV Seminar*. Birkhäuser Verlag, Basel.
- GROENEBOOM, P. and WELLNER, J. A. (2001). Computing Chernoff's distribution. *J. Comput. Graph. Statist.* **10** 388–400.
- HARGÉ, G. (2004). A convex/log-concave correlation inequality for Gaussian measure and an application to abstract Wiener spaces. *Probab. Theory Related Fields* **130** 415–440.
- HARRISON, P. G. (1990). Laplace transform inversion and passage-time distributions in Markov processes. *J. Appl. Probab.* **27** 74–87.
- HUANG, J. and WELLNER, J. A. (1995). Estimation of a monotone density or monotone hazard under random censoring. *Scand. J. Statist.* **22** 3–33.
- HUANG, Y. and ZHANG, C.-H. (1994). Estimating a monotone density from censored observations. *Ann. Statist.* **22** 1256–1274.
- JANSON, S., LOUCHARD, G. and MARTIN-LÖF, A. (2010). The maximum of Brownian motion with parabolic drift. *Electronic Journal of Probability* **15** 1893–1929.
- KARLIN, S. (1968). *Total positivity. Vol. I.* Stanford University Press, Stanford, Calif.
- KIM, J. and POLLARD, D. (1990). Cube root asymptotics. *Ann. Statist.* **18** 191–219.
- LE CAM, L. (1986). The central limit theorem around 1935. *Statist. Sci.* **1** 78–96. With comments, and a rejoinder by the author.
- LEURGANS, S. (1982). Asymptotic distributions of slope-of-greatest-convex-minorant estimators. *Ann. Statist.* **10** 287–296.
- MARSHALL, A. W. and OLKIN, I. (1979). *Inequalities: theory of majorization and its applications*, vol. 143 of *Mathematics in Science and Engineering*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York.
- MARSHALL, A. W., OLKIN, I. and ARNOLD, B. C. (2011). *Inequalities: theory*

- of majorization and its applications. 2nd ed. Springer Series in Statistics, Springer, New York.
- MERKES, E. P. and SALMASSI, M. (1997). On univalence of certain infinite products. *Complex Variables Theory Appl.* **33** 207–215.
- OLKIN, I. and TONG, Y. L. (1988). Peakedness in multivariate distributions. In *Statistical decision theory and related topics, IV, Vol. 2 (West Lafayette, Ind., 1986)*. Springer, New York, 373–383.
- OLVER, F. W. J., LOZIER, D. W., BOISVERT, R. and CLARK, C. W. (2010). *NIST Handbook of Mathematical Functions*. Cambridge University Press.
- PRAKASA RAO, B. L. S. (1969). Estimation of a unimodal density. *Sankhyā Ser. A* **31** 23–36.
- PRAKASA RAO, B. L. S. (1970). Estimation for distributions with monotone failure rate. *Ann. Math. Statist.* **41** 507–519.
- PROSCHAN, F. (1965). Peakedness of distributions of convex combinations. *Ann. Math. Statist.* **36** 1703–1706.
- ROCKAFELLAR, R. T. and WETS, R. J.-B. (1998). *Variational analysis*, vol. 317 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin.
- SALMASSI, M. (1999). Inequalities satisfied by the Airy functions. *J. Math. Anal. Appl.* **240** 574–582.
- SCHOENBERG, I. J. (1951). On Pólya frequency functions. I. The totally positive functions and their Laplace transforms. *J. Analyse Math.* **1** 331–374.
- SHORACK, G. R. (2000). *Probability for statisticians*. Springer Texts in Statistics, Springer-Verlag, New York.
- VAN EEDEN, C. (1957). Maximum likelihood estimation of partially or completely ordered parameters. I. *Nederl. Akad. Wetensch. Proc. Ser. A.* **60** = *Indag. Math.* **19** 128–136.
- VAN ZWET, W. R. (1964a). Convex transformations: A new approach to skewness and kurtosis. *Statistica Neerlandica* **18** 433–441.
- VAN ZWET, W. R. (1964b). *Convex transformations of random variables*, vol. 7 of *Mathematical Centre Tracts*. Mathematisch Centrum, Amsterdam.